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Computing the Integrated Circumradius and Area Formulae for Cyclic Heptagons by Numerical Interpolation *

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Abstract

This paper describes computations of the relations between the circumradius R and area S of cyclic polygons given by the lengths of the sides. The classic results of Heron and Brahmagupta clearly show that the product of R and S is expressed by the lengths of the sides for triangles and cyclic quadrilaterals. In a previous paper by the author published in 2015, the similar *integrated formulae* of the circumradius and the area for cyclic pentagons and hexagons were explicitly computed, where elimination by resultants and factorization of polynomials were minutely applied. In this study, extending the previous results, we computed the integrated formulae for cyclic heptagons. However, we adopted the method of numerical interpolation instead of elimination, because it is almost impossible to compute the resultants for the heptagon case. As a result, we succeeded in computing the integrated formula, which is a polynomial equation in $z = 4SR$ with degree 38 and 31,590 terms. This polynomial is straightforwardly transformed into a polynomial in $Z = (4SR)^2$ with degree 38 and 973,558 terms, which is supposed to be the substitution of the side length $a_8 = 0$ into the integrated formula for cyclic octagons, if we could have its explicit expression.

1 Introduction

In this study, we consider a classic problem in Euclidean geometry for cyclic polygons; that is, n-gons inscribed in a circle, given by the lengths of sides $a_1, a_2, \ldots, a_n$. Since Robbins [9] discovered the area formula for cyclic pentagons in 1994, area formulae for cyclic n-gons, up to $n = 7, 8$, have been mainly studied by several authors ([1], [2], [8], [10], [11]). On the other hand, the author of the present paper has been clarifying the circumradius formulae for cyclic heptagons and octagons in ([3], [5], [6]).

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Compared with these studies, the relation between the area and circumradius has seldom been discussed except for ([10], [4]). In a previous paper by the author [4], the relation between the circumradius R and the area S for cyclic pentagons and hexagons was specifically addressed. As a result, we succeeded in computing the *integrated formulae* of R and S explicitly. Based on these results, the present paper focuses on the computation of the integrated formulae for cyclic heptagons. It seems that only a slide presentation by Svrtan [11] has reported the definition polynomial in $Z = (4SR)^2$ as having 31,590 terms. Even though $(4SR)^2$ is presumed to be the error in $4SR$, there is almost no description about the relevant algorithm in [11]. Since Svrtan mainly discusses the area formulae ($n = 7, 8$), it is difficult to reproduce his results. Therefore, in this paper, we show the details of our algorithm by numerical interpolation, in order to verify whether the results match.

In Section 2, we review the classical results for triangles and cyclic quadrilaterals in our notation and formulation. In Section 3, we cite the results of the author's previous paper [4] for cyclic pentagons and hexagons.

In Section 4, we describe the details for the computation of the integrated formulae for cyclic heptagons. In this step, the "new Brahmagupta's formula" discovered by Svrtan [11] is applied. In Section 5, we show our algorithm by numerical interpolation in detail. Certainly the same result with 31,590 terms is obtained. Finally, we summarize the results of this study and discuss the extension of the formulae to cyclic octagons in Section 6.

2 Classical results for n-gons ($n = 3, 4$)

Firstly, for a triangle with side lengths a_1, a_2, and a_3, the classic formula derived by Heron gives its circumradius and area as follows:

$$\begin{cases} R &= \dfrac{a_1a_2a_3}{\sqrt{(a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3)}}, \\ S &= \dfrac{\sqrt{(a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3)}}{4}. \end{cases} \tag{1}$$

It is straightforward to combine these equations, and we obtain the relation

$$4SR = a_1a_2a_3. \tag{2}$$

We should note that, in our formulation, the area of the triangle between $\vec{OA} = [x_1, y_1]$ and $\vec{OB} = [x_2, y_2]$ is defined as the determinant

$$S = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \tag{3}$$

whose sign depends on the direction of the angle between these two vectors. Hence, discarding the sign of area S of polygons, the formula for triangles given by Eq. (2) is rewritten as

$$\begin{cases} |z| - \sqrt{s_3} &= 0, \\ Z - s_3 &= 0, \end{cases} \tag{4}$$

where $z = 4SR$, $Z = (4SR)^2$, $s_3 = a_1^2a_2^2a_3^2$, and $\sqrt{s_3} = a_1a_2a_3$ using elementary symmetric polynomials with a_i^2's.

Secondly, Brahmagupta's formula gives the circumradius and area of a cyclic quadrilateral, and it is again straightforward to integrate this into

$$(4SR)^2 = (a_1a_2 + a_3a_4)(a_1a_3 + a_2a_4)(a_1a_4 + a_2a_3). \tag{5}$$

Using the elementary symmetric polynomials with a_i^2's of the 4th order, this equation is rewritten in reduced form as

$$Z = s_3 + s_1\sqrt{s_4}, \tag{6}$$

where $s_1 = a_1^2 + \cdots + a_4^2$, $s_2 = a_1^2 a_2^2 + \cdots$, and $\sqrt{s_4} = a_1 a_2 a_3 a_4$.

Since Eq. (5) represents the case of convex quadrilaterals, the other case of non-convex, crossing figures is given by letting $a_4 := -a_4$, as follows:

$$(4SR)^2 = -(a_1a_2 - a_3a_4)(a_1a_3 - a_2a_4)(a_1a_4 - a_2a_3). \tag{7}$$

Converting this equation into an expression by elementary symmetric polynomials, we obtain

$$Z = s_3 - s_1\sqrt{s_4}. \tag{8}$$

We should note that a good insight into the structure of the formulae is provided by the introduction of an auxiliary expression $\sqrt{s_n} = a_1 \cdots a_n$, as well as the notion of *crossing parity* ε ([9], [2]), where ε is 0 for a triangle, 1 for a convex quadrilateral, and -1 for a non-convex quadrilateral.

Theorem 1
In conclusion, combining Eqs. (4) (6) (8) into a polynomial form, we have

$$\begin{cases} \varphi_3(z) &= |z| - \sqrt{s_3}, \\ \psi_{3,4}(Z) &= Z - (s_3 + \varepsilon \cdot s_1\sqrt{s_4}), \end{cases} \tag{9}$$

as defining polynomials in z and Z for triangles and cyclic quadrilaterals.

Since we have $s_3^{(3)} = s_3^{(4)}\,|_{a_4=0}$ and so on, the notations are intentionally combined for the cases $\varepsilon = 0, \pm 1$. Hereafter, we abbreviate $s_i^{(n)}$ simply as s_i, if the order n is obvious in the context.

3 Latest results for n-gons ($n = 5, 6$)

In the author's previous paper [4], we succeeded in specifying the structure of integrated formulae for cyclic n-gons ($n = 5, 6$) in detail. First, we derived the following polynomial equation by dividing the cyclic pentagon into a triangle and a cyclic quadrilateral.

Theorem 2
The defining polynomial in $z = 4SR$ for cyclic pentagons is given by

$$\begin{aligned} \varphi_5(z) = {} & |z|^7 - 2s_3|z|^5 - (s_1^2 + 4s_2)\sqrt{s_5}|z|^4 + (s_3^2 - s_1^2 s_4 - 14 s_1 s_5)|z|^3 \\ & -(s_1^2 s_3 + 8 s_1 s_4 - 4 s_2 s_3 + 24 s_5)\sqrt{s_5}|z|^2 \\ & -(s_1^2 s_2 - 4 s_2^2 + 2 s_1 s_3 + 16 s_4) s_5 |z| \\ & -(s_1^3 - 4 s_1 s_2 + 8 s_3) s_5 \sqrt{s_5} \qquad \textit{(18 terms).} \end{aligned} \tag{10}$$

Rewriting the equation $\varphi_5(z) = 0$ by the terms with even degrees and odd degrees as

$$|z|\left(z^6 - 2s_3 z^4 + \cdots\right) = (s_1^2 + 4s_2)\sqrt{s_5} z^4 + \cdots + (s_1^3 - 4s_1 s_2 + 8s_3) s_5 \sqrt{s_5}, \tag{11}$$

squaring both sides, and substituting $z^2 = Z$, we obtain the polynomial in $Z = z^2$.

Theorem 3
The defining polynomial in $Z = (4SR)^2$ for cyclic pentagons has the following form:

$$\begin{aligned} \psi_5(Z) &= Z^7 - 4s_3Z^6 + \left(-28s_1s_5 - 2s_1^2s_4 + 6s_3^2\right)Z^5 + \cdots \\ &\quad \cdots - (s_1^3 - 4s_1s_2 + 8s_3)^2 s_5^3 \qquad \textit{(63 terms)}. \end{aligned} \tag{12}$$

Next, we computed the case of a convex cyclic hexagon by dividing it into two cyclic quadrilaterals. Using elimination by resultants and polynomial factorization, we obtained the following theorem and corollaries.

Theorem 4
One of the defining polynomials of $Z = (4SR)^2$ for cyclic hexagons has the following form:

$$\begin{aligned} \psi_6^{(+)}(Z) &= Z^7 - (4s_3 + 28\sqrt{s_6})Z^6 + (\cdots)Z^5 + \cdots + (\cdots)Z \\ &\quad -(s_1^3 - 4s_1s_2 + 8s_3 - 16\sqrt{s_6})^2 \\ &\quad \times(s_5^3 - 4\sqrt{s_6}^5 + (s_1^3 - 4s_1s_2 + 4s_3)\sqrt{s_6}^4 \\ &\qquad +(-s_1^2s_4 + 2s_1s_5 + 4s_2s_4 - s_3^2)\sqrt{s_6}^3 + (s_1s_3s_5 - 4s_4s_5)\sqrt{s_6}^2 \\ &\qquad -s_2s_5^2\sqrt{s_6}) \qquad \textit{(327 terms)}. \end{aligned} \tag{13}$$

Corollary 5
(i) If we replace $\sqrt{s_6}$ with $-\sqrt{s_6}$ in $\psi_6^{(+)}(Z)$, we obtain the other polynomial $\psi_6^{(-)}(Z)$, which corresponds to the group that does not include the convex cyclic hexagon.
(ii) If we replace $\sqrt{s_6}$ with 0 in $\psi_6^{(+)}(Z)$ and $\psi_6^{(-)}(Z)$, we obtain the pentagon formula $\psi_5(Z)$ in Eq. (12). That is, these three polynomials are represented uniformly through the crossing parity ε.

Theorem 6
In conclusion, combining Eqs. (10) (12) (13) into a polynomial form, we have

$$\begin{cases} \varphi_5(z) = |z|^7 - 2s_3|z|^5 - (s_1^2 + 4s_2)\sqrt{s_5}|z|^4 + \cdots, \\ \psi_{5,6}(Z) = Z^7 - (4s_3 + 28\varepsilon\sqrt{s_6})Z^6 + \cdots, \end{cases} \tag{14}$$

where $\varepsilon = 0$ for cyclic pentagons, $\varepsilon = 1$ for the group that includes convex cyclic hexagons, and $\varepsilon = -1$ for the other group.

Therefore, the final goal of the present study is to find *integrated formulae* for cyclic heptagons and octagons analogous to Eq. (14). Comparing the area formulae and circumradius formulae, we can speculate that the relations between S and R are expressed by the polynomials in $z = 4SR$ for $n = 7$ and $Z = (4SR)^2$ for $n = 7, 8$, with degree 38. As a result of this study, we succeeded in computing such formulae for cyclic heptagons explicitly as speculated.

4 Main results for cyclic heptagons

We have succeeded in showing that the products of area S and circumradius R of cyclic heptagons $z = 4SR$ and $Z = (4SR)^2$ are the respective roots of the following polynomials:

$$\begin{cases} \varphi_7(z) = |z|^{38} - 8s_3|z|^{36} + \cdots + B_1z + B_0 \qquad (31,590 \text{ terms}), \\ \psi_7(Z) = Z^{38} - 16s_3Z^{37} + \cdots + C_1Z + C_0 \qquad (973,558 \text{ terms}). \end{cases} \tag{15}$$

Here, the respective coefficients belong to $B_j \in \mathbf{Z}[s_1, \ldots, s_6, \sqrt{s_7}]$), and $C_j \in \mathbf{Z}[s_1, \ldots, s_6, s_7]$), where s_i denotes the elementary symmetric polynomials of the 7th degree with a_j^2, as follows:

$$s_1 = a_1^2 + a_2^2 + \cdots + a_7^2, \quad \ldots, \quad \text{and} \quad s_7 = a_1^2 a_2^2 \cdots a_7^2, \quad (\sqrt{s_7} = a_1 a_2 \cdots a_7). \tag{16}$$

The precise forms including the number of terms in each coefficient are shown in Table 2.

4.1 Constant terms C_0 in $\psi_7(Z)$ and B_0 in $\varphi_7(z)$

We assume that the area formulae and circumradius formulae for $n = 7, 8$ are already computed. Expanding the area formula for $x = (4S)^2$ by Maley et al. [2], we obtain

$$\begin{cases} G_7(x) &= x^{38} + M'_{37}x^{37} + \cdots + M'_0 \qquad \text{(955,641 terms)}, \\ G_8(x) &= x^{38} + M_{37}x^{37} + \cdots + M_0 \qquad \text{(3,248,266 terms)}, \\ & \qquad (M_i \in \mathbf{Z}[s_1, \ldots, s_7, \varepsilon\sqrt{s_8}], \ M'_i = M_i|_{\varepsilon=0}). \end{cases} \tag{17}$$

As elucidated in ([5], [6]), the circumradius formulae for $y = R^2$ are expressed as follows:

$$\begin{cases} F_7(y) &= P'_{38}y^{38} + \cdots + P'_1 y + P'_0 \qquad \text{(199,695 terms)}, \\ F_8(y) &= P_{38}y^{38} + \cdots + P_1 y + P_0 \qquad \text{(845,027 terms)}, \\ & \qquad (P_i \in \mathbf{Z}[s_1, \ldots, s_7, \varepsilon\sqrt{s_8}], \ P'_i = P_i|_{\varepsilon=0}). \end{cases} \tag{18}$$

Combining Eqs. (17) and (18), the constant term C_0 in Eq. (15) is straightforwardly computed. Let each root of $G_7(x)$ and $F_7(y)$ be x_i and y_i, respectively. Since we have $x = (4S)^2$ and $y = R^2$, the constant term of the polynomial with roots $Z = xy$ is given by

$$\prod_{i=1}^{38} (x_i y_i) = \left(\prod x_i\right) \cdot \left(\prod y_i\right) = M'_0 \cdot \frac{P'_0}{P'_{38}} = C_0, \tag{19}$$

where M'_0 is divisible by P'_{38}, and we have the polynomial expression $C_0 \in \mathbf{Z}[s_1, \ldots, s_6, s_7]$.

Moreover, this polynomial is factorized in $\mathbf{Z}[s_1, \ldots, s_6, \sqrt{s_7}]$ as $C_0 = (\pm B_0)^2$. Here, the plus or minus sign is decided by the numerical substitution of $a_i := p_i$ with random primes, and we obtain the constant term B_0 in Eq. (15).

4.2 Extracting underlying relational expressions

4.2.1 New Brahmagupta's formula

In the author's preceding paper [7], a cyclic heptagon is divided into a hexagon and a triangle. In contrast, here we apply the "new Brahmagupta's formula" discovered by Svrtan [11], which leads to simpler expressions of geometric relations.

Brahmagupta's area formula for a (convex) cyclic quadrilateral with side lengths $\{a, b, c, d\}$ has the following form:

$$16S^2 = 2\left(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2\right) - a^4 - b^4 - c^4 - d^4 + 8abcd. \tag{20}$$

We express the right-hand side as $G(d)$, which we regard as a function of d. Letting $g(a, b, c;\ d) = G'(d)/4$, we have

$$g(a, b, c;\ d) = -d^3 + (a^2 + b^2 + c^2)d + 2abc. \tag{21}$$

Dividing a (convex) cyclic heptagon $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ by a diagonal d into quadrilaterals $\{a_1, a_2, a_3, d\}$ and $\{a_4, a_5, a_6, d\}$, and letting the area of each be S_1 and S_2, respectively, we have, according to Svrtan [11],

$$\frac{S_2}{S_1} = -\frac{g(a_4, a_5, a_6;\ d)}{g(a_1, a_2, a_3;\ d)}, \tag{22}$$

which has a more compact form than S_2^2/S_1^2 obtained by Eq. (20).

We note that Eq. (20) contains a triangle case (for example, let $a = 0$), and Eq. (22) also holds for the division of a cyclic pentagon.

4.2.2 Application to the division of a cyclic heptagon

We consider a given cyclic heptagon $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ with area S and circumradius R that is divided by diagonals d_1, d_2 into a quadrilateral $\{a_1, a_2, a_3, d_1\}$, a triangle $\{d_1, d_2, a_7\}$, and a quadrilateral $\{a_4, a_5, a_6, d_2\}$. If we let the area of each be S_1, S_2, and S_3, respectively, then we have $S = S_1 + S_2 + S_3$.

Applying Eq. (22), we abbreviate each rational expression as follows:

$$\frac{S_2}{S_1} = -\frac{g(a_7, d_2, 0;\ d_1)}{g(a_1, a_2, a_3;\ d_1)} = -\frac{\alpha}{\beta}, \qquad \frac{S_2}{S_3} = -\frac{g(a_7, d_1, 0;\ d_2)}{g(a_4, a_5, a_6;\ d_2)} = -\frac{\gamma}{\delta}. \tag{23}$$

If we substitute $S_1 = -\dfrac{\beta}{\alpha} S_2$ and $S_3 = -\dfrac{\delta}{\gamma} S_2$ into $S = S_1 + S_2 + S_3$, then we have

$$S = \left(-\frac{\beta}{\alpha}\right) S_2 + S_2 + \left(-\frac{\delta}{\gamma}\right) S_2. \tag{24}$$

Multiplying by $4R$ on both sides and letting $z = 4SR$, we have

$$z = \left(-\frac{\beta}{\alpha} + 1 - \frac{\delta}{\gamma}\right) 4S_2 R. \tag{25}$$

Since we have $4S_2R = d_1 d_2 a_7$ from Eq. (2), clearing the denominators, we obtain

$$\alpha\gamma \cdot z = (-\beta\gamma + \alpha\gamma - \alpha\delta)\, d_1 d_2 a_7. \tag{26}$$

Moreover, we have $d_1 \mid \alpha$ and $d_2 \mid \gamma$, and dividing both sides by $d_1 d_2$, we obtain

$$\begin{aligned} f_0(a_i, d_1, d_2, z) \quad &= \quad \left(a_7^4 - \left(d_1^2 - d_2^2\right)^2\right) \cdot z + (\beta\gamma - \alpha\gamma + \alpha\delta)\, a_7 \\ &\qquad (\alpha, \beta, \gamma, \delta \ \in \ \mathbf{Z}[a_1, a_2, a_3, a_4, a_5, a_6, a_7, d_1, d_2])\,. \end{aligned} \tag{27}$$

This polynomial equation is linear with $z(= 4SR)$ for a cyclic heptagon and it has only 31 terms in the expanded form. In addition, the polynomial f_0 has degree 4 both in d_1 and d_2.

4.2.3 Elimination of diagonals d_1, d_2

In order to eliminate d_1, d_2 from $f_0(a_i, d_1, d_2, z)$ in Eq. (27), we need two independent polynomials in $\mathbf{Z}[a_i, d_1, d_2]$. Applying the circumradius formula ($n = 3, 4$) to each part, we have the following three polynomials:

$$\begin{aligned} f_1(a_1, a_2, a_3, d_1, R) &= \left(a_1^4 + a_2^4 + a_3^4 + d_1^4 + \cdots\right) R^2 + \left(a_1^2 a_2^2 a_3^2 + a_1^2 a_2^2 d_1^2 + a_1^2 a_3^2 d_1^2 + a_2^2 a_3^2 d_1^2 + \cdots\right), \\ f_2(a_7, d_1, d_2, R) &= \left(a_7^4 + d_1^4 + d_2^4 - 2a_7^2 d_1^2 - 2a_7^2 d_2^2 - 2d_1^2 d_2^2\right) R^2 + d_1^2 d_2^2 a_7^2, \\ f_3(a_4, a_5, a_6, d_2, R) &= \left(a_4^4 + a_5^4 + a_6^4 + d_2^4 + \cdots\right) R^2 + \left(a_4^2 a_5^2 a_6^2 + a_4^2 a_5^2 d_2^2 + a_4^2 a_6^2 d_2^2 + a_5^2 a_6^2 d_2^2 + \cdots\right). \end{aligned} \tag{28}$$

First, we eliminate R^2 by the resultant as follows:

$$\begin{aligned} &h_1(a_1,\ldots,a_6,d_1,d_2) := \mathrm{Res}_{R^2}(f_1,f_3) \qquad (176 \text{ terms}, \ \deg_{d_1} h_1 = 4, \ \deg_{d_2} h_1 = 4),\\ &h_2(a_4,a_5,a_6,a_7,d_1,d_2) := \mathrm{Res}_{R^2}(f_2,f_3) \qquad (52 \text{ terms}, \ \deg_{d_1} h_2 = 4, \ \deg_{d_2} h_2 = 7).\end{aligned} \tag{29}$$

Next, we eliminate d_2 for $\{h_1,h_2\}$ and $\{h_1,f_0\}$, remove the content part if it exists, and we have

$$\begin{aligned} &u_1(a_i,d_1) := \mathrm{PrimitivePart}(\mathrm{Res}_{d_2}(h_1,h_2),d_1) \qquad (1,060,738 \text{ terms} \ \ \deg_{d_1} u_1 = 38),\\ &u_2(a_i,d_1,z) := \mathrm{Res}_{d_2}(h_1,f_0) \qquad (2,404,502 \text{ terms} \ \ \deg_{d_1} u_2 = 28, \ \deg_z u_2 = 4).\end{aligned} \tag{30}$$

Unfortunately, eliminating d_1 from $\{u_1,u_2\}$ is not realistic, because of their sizes. Speculating under the numerical substitution of $a_i := p_i$ with random primes, we should have the following factorization:

$$\begin{aligned} v(a_i,z) &:= \mathrm{Res}_{d_1}(u_1,u_2)\\ &= \left(z^{38}+\triangledown z^{36}+\cdots\right)\left(\triangle z^{38}+\triangle z^{37}+\cdots\right)\left(\square z^{38}+\square z^{37}+\cdots\right)\left(\diamond z^{38}+\diamond z^{37}+\cdots\right).\end{aligned} \tag{31}$$

Here, $w(a_i,z) = z^{38}+\triangledown z^{36}+\cdots$ is the polynomial in $\mathbf{Z}[a_1,\ldots,a_7][z]$ with 45,728,577 terms, which is the expanded form of $\varphi_7(z) \in \mathbf{Z}[s_1,\ldots,s_6,\ \sqrt{s_7}][z]$ in Eq. (15). Therefore, we apply a numerical interpolation method to Eq. (31), instead of symbolic computation of the resultant.

5 Numerical interpolation method

In computing the circumradius formula for cyclic octagons, the author [6] used a numerical interpolation method together with resultant computation. In this study, we apply similar numerical algorithms to the case of the integrated formulae for cyclic heptagons.

5.1 Analysis of the distribution of total degrees

We define the total degree of a power product in a_i^2's as follows:

$$\text{t-deg}\left(a_1^{2k_1}a_2^{2k_2}\cdots a_n^{2k_n}\right) := k_1+k_2+\cdots+k_n. \tag{32}$$

Since elementary symmetric polynomials $s_1 = a_1^2+\cdots+a_n^2$, $s_2 = a_1^2a_2^2+\cdots$, $\ldots$, $s_n = a_1^2a_2^2\cdots a_n^2$ are homogeneous, with each having degree i ($i = 1,\ldots,n$), we have

$$\text{t-deg}\left(s_1^{k_1}s_2^{k_2}\cdots s_n^{k_n}\right) = k_1+2k_2+\cdots+nk_n. \tag{33}$$

Exceptionally, we define t-deg($\sqrt{s_7}$) = 7/2, where $\sqrt{s_7} = a_1a_2\cdots a_7$.

By analogy with the cases of the area formula and circumradius formula, we can speculate the distribution of total degrees in the integrated formulae, as shown (in parentheses) in Table 1.

Next, we search for all the 7-tuples of integers $(e_1,\ldots,e_6,e_7)$ so that $e_1+2e_2+\cdots+6e_6+3.5e_7 = d$ $(d = 1.5,3,\ldots,55.5)$ is satisfied. We let the numbers of found 7-tuples be N_ν $(\nu = 37,\ldots,1)$, which are shown in the "#candidates" column in Table 2.

Degree in main variable	38	37	36	$\cdots$	$\cdots$	2	1	0
Area ($x = (4S)^2$)	0	2	4	$\cdots$	$\cdots$	72	74	76
Circumradius ($y = R^2$)	32	33	34	$\cdots$	$\cdots$	68	69	70
Integrated ($z = 4SR$)	0	(1.5)	(3)	$\cdots$	$\cdots$	(54.0)	(55.5)	57.0
Integrated ($Z = (4SR)^2$)	0	(3)	(6)	$\cdots$	$\cdots$	(108)	(111)	114

Table 1: Total degree in each coefficient for cyclic heptagon formulae

5.2 Determination of coefficients

We apply the following steps for each v (= 36, ..., 1) to compute the coefficients with total degree $d_v(= 3, \ldots, 55.5)$ in Table 1 by numerical interpolation. It is obvious that $N_{37} = 0$ for $d_{37} = 1.5$ because $e_1 + 2e_2 + \cdots + 6e_6 + 3.5e_7 = 1.5$ has no non-negative integer solution.

(1) We generate N_v monomials $m_k = s_1^{e_1^{(k)}} \cdots s_6^{e_6^{(k)}} \sqrt{s_7}^{e_7^{(k)}}$ $(k = 1, \ldots, N_v)$.
(2) We let $f(a_1, \ldots, a_7) = c_1 m_1 + \cdots + c_{N_v} m_{N_v}$ using indeterminate coefficients $c_1, \ldots, c_{N_v}$.
(3) We choose a set of random prime numbers $(p_1, \ldots, p_7)$ and substitute them into $f(a_i)$. On the other hand, we compute $w(p_i, z)$ according to Eq. (31), and extract the coefficient t_v of z^v. Then, we have a linear equation over the integers $f(p_i) = t_v$ with indeterminates $c_1, \ldots, c_{N_v}$.
(4) If we choose "linearly independent" N_v sets of 7-tuples, we have a system of linear equations over the integers $A\boldsymbol{c} = \boldsymbol{t}$. Solving this equation, we obtain the coefficients $\boldsymbol{c} = (c_1, \ldots, c_{N_v})^T$.

5.3 Devices for improvement of efficiency

In the actual implementation, we applied the following techniques to improve efficiency.

(1) First, we searched all the candidate monomials for $d = 1.5, 3, \ldots, 55.5$, and obtained the numbers shown in the "#candidates" column in Table 2. That is, we assumed first of all that the maximum number of N_v was 26,226 for $v = 2$ (the coefficient of z^2).
(2) Next, in order to obtain "linearly independent" evaluation points, we generated the sequence of prime numbers

$$(a_1, a_2, \ldots, a_7) = (101, 103, \ldots, 131),\ (103, 107, \ldots, 137),\ \ldots,$$

and considered heptagons with side length of these prime numbers. We computed 26,226 definition polynomials $w(p_i, z)$, using the resultant and factorization shown in Eq. (31). We saved all of these polynomials $w_1(p_1, \ldots, p_7, z), \ldots, w_{26,226}(p_{26,226}, \ldots, p_{26,232}, z)$, and used as many as needed for computing each coefficient of z^v.
(3) Solving a linear equation $A\boldsymbol{c} = \boldsymbol{t}$ over $\mathbf{Z}$ directly with a large matrix size such as 26,226 is almost impossible. Instead, we solved the equation $A\boldsymbol{c} = \boldsymbol{t}$ over $\mathbf{Z}_p$ and computed the solution, such as

$$\boldsymbol{c} = [\ldots, \star, 0, \star, \ldots, \star, 0, 0, \ldots]^T \pmod{p}.$$

Then, we extracted the non-zero elements ($\star$'s) and the matrix size was reduced, for example, to 664 in the case of z^2. Even though we confirmed that $A \mod p$ is regular, we note that this step is probabilistic.
(4) Finally, we solved the equation with reduced size $A'\boldsymbol{c}' = \boldsymbol{t}'$ over $\mathbf{Z}$, and checked the solution by substitution into $A\boldsymbol{c} = \boldsymbol{t}$ over $\mathbf{Z}$. Eventually, the maximum size of A' was 2,504 for the coefficient of z^{10}, as shown in the "#terms of φ_7" column in Table 2.

6 Concluding remarks and extension to cyclic octagons

In this study, we succeeded in computing the integrated circumradius and area formula for cyclic heptagons $\varphi_7(z)$ in Eq. (15) by numerical interpolation. In order to convert the polynomial equation $\varphi_7(z) = |z|^{38} - 8s_3|z|^{36} + \cdots = 0$ into the equation in $Z\left(= z^2 = (4SR)^2\right)$, we separate it into the terms with even degrees and odd degrees, as follows:

$$|z|\left(B_{35}|z|^{34} + \cdots + B_1\right) = |z|^{38} - s_3|z|^{36} + \cdots + B_0. \tag{34}$$

Squaring both sides and substituting $z^2 = Z$, we obtain the other polynomial $\psi_7(Z)$ in Eq. (15).

In these processes, we needed about 12.5 days of CPU time in total for computing Eq. (31) for 26,226 patterns of heptagons in our environment: Maple 2017 on Win64, Xeon (2.93 GHz) $\times$ 2, 192 GB RAM. In contrast, it took about 3.9 days of CPU time in total to solve all the systems of linear equations for the undetermined variables.

The number of terms of $\varphi_7(z)$, 31,590, is identical to that reported by Svrtan [11]. Since Svrtan's elimination algorithm is unknown, our results correspond to the validation of preceding studies, and we believe it significant to have clarified our algorithm and its cost concretely.

The next goal should be the octagon formula $\psi_8^{(\pm)}(Z)$, which satisfies $\psi_7(Z) = \psi_8^{(\pm)}(Z)|_{\varepsilon=0.}$ We have applied similar methods of numerical interpolation, but the latest result is

$$\begin{aligned} \psi_8^{(+)}(Z) &= Z^{38} - 16s_3Z^{37} + D_{36}Z^{36} + \cdots + D_{18}Z^{18} + \left(D_{17}Z^{17} + \cdots + D_1Z\right) + D_0, \\ & \qquad D_i \in [s_1, \ldots, s_7, \varepsilon\sqrt{s_8}] \quad (\varepsilon = 1). \end{aligned} \tag{35}$$

The coefficient D_{18} has 77,131 terms, which is the largest among those already computed. Exceptionally, the constant term D_0 is computed by an analogous process to that of Eq. (19), and it has 554,173 terms. In contrast, the terms in the parentheses $D_{17}, \ldots, D_1$ are unable to be computed at present.

Since the matrix size for D_{17} is $125,054$ and this increases to $4,116,544$ for D_1, it is impossible to compute these coefficients in the present computational environment described above. It seems that we need to find another principle for elimination algorithms.

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deg in z	t-deg	#candidates	#terms of φ_7	deg in Z	t-deg	#terms of ψ_7
0	57.0	(not used)	295	0	114	5,120
1	55.5	21,873	465	1	111	9,577
2	54.0	26,226	664	2	108	15,564
3	52.5	16,475	926	3	105	23,239
4	51.0	19,928	1,230	4	102	32,597
5	49.5	12,241	1,551	5	99	43,316
6	48.0	14,950	1,814	6	96	54,102
7	46.5	8,946	2,075	7	93	64,045
8	45.0	11,044	2,237	8	90	72,291
9	43.5	6,430	2,392	9	87	78,269
10	42.0	8,033	2,504	10	84	81,969
11	40.5	4,526	2,163	11	81	77,990
12	39.0	5,731	2,258	12	78	71,316
13	37.5	3,120	1,758	13	75	63,500
14	36.0	4,011	1,845	14	72	55,553
15	34.5	2,093	1,309	15	69	47,257
16	33.0	2,738	1,376	16	66	39,733
17	31.5	1,367	897	17	63	32,591
18	30.0	1,824	969	18	60	26,301
19	28.5	860	591	19	57	20,757
20	27.0	1,175	632	20	54	16,064
21	25.5	522	359	21	51	12,152
22	24.0	733	389	22	48	9,063
23	22.5	300	211	23	45	6,636
24	21.0	436	226	24	42	4,776
25	19.5	164	116	25	39	3,366
26	18.0	248	123	26	36	2,328
27	16.5	82	60	27	33	1,561
28	15.0	131	62	28	30	1,025
29	13.5	38	28	29	27	645
30	12.0	65	30	30	24	393
31	10.5	15	11	31	21	227
32	9.0	28	12	32	18	124
33	7.5	5	5	33	15	63
34	6.0	11	4	34	12	30
35	4.5	1	1	35	9	12
36	3.0	3	1	36	6	4
37	1.5	0	0	37	3	1
38	0.0	1	1	38	0	1

Table 2: Each coefficient in the heptagon formulae $\varphi_7(s_i;\ z)$ and $\psi_7(s_i;\ Z)$